DYNAMIC PROBLEM OF THE DIE ON AN ELASTIC HALF-PLANE

(DYNAMICHESKAIA ZADACHA O SHTAMPE NA UPROGOI POLUPLOSKOSTI)

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The mixed boundary value problem (the problem of the die) in the static theory of elasticity is well studied [1-3]. In transient dynamics of an elastic medium for a half-space, the problems were studied effectively for which, for certain initial conditions, there are given on the boundary either the displacements of the tensions [4-6] or certain components of displacement and certain components of tension [4]. Paper [8] considered Lamb's problem for mixed boundary conditions, permitting to investigate the propagation of longitudinal and transverse waves.

In the present article we consider the plane case of the mixed boundary value problem for the dynamic equations of the theory of elasticity.

1. We consider the equation

$$(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+\mu\Delta\mathbf{u}-\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=0 \tag{1.1}$$

Here $\mathbf{u} = (u, v)$ is the displacement vector; λ, μ are Lamé's coefficients, ρ is the density and the elastic body occupies the half-plane $y \geqslant 0$. The initial conditions, for the sake of simplicity, are taken to be zero,

$$\mathbf{u}=0, \quad \frac{\partial\mathbf{u}}{\partial t}=0 \quad \text{at} \quad t=0$$

and the boundary conditions are:

$$\sigma_{xy}=0, \quad \sigma_{yy}=0 \text{ where } |x|>l, \qquad v=f(x,t) \text{ where } |x|<l \qquad \text{for } y=0 \tag{1.2}$$

here σ_{yy}, σ_{xy} are components of the stress tensor and $f(x, t)$ is a given function which is assumed to be bounded and possesses a finite number of lines of discontinuity.

The physical meaning of the boundary conditions is as follows.

A rigid die is impressed without friction into the elastic half-plane along the portion $|x| \leqslant l$; it produces on this portion the displacement $v = f(x, t)$, while the remaining part of the boundary $|x| > l$ is free of tension.

Paper [9] contains the solution of an analogous problem for one wave equation. It is a pity that the author was not acquainted with the work of Galin [13], where a problem is solved which may be reduced to the one of [9], and therefore [9] does not contain the proper reference. This drawback is corrected in the present work.

The aim of the present paper is the construction for $y = 0$ of the values σ_{yy} for $|x| < l$, that is of the stresses underneath the die, and in addition of v for $|x| > l$, that is of the vertical displacement on the free boundary. The construction of these functions on the boundary represents, above all, an immediate interest and, in addition, reduces our problem to one of the solved boundary value problems of an elastic half-plane [4-7].

As is known [4], any solution of equations (1.1) may be represented in the plane case in the form

$$u = \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial x} \tag{1.3}$$

where the functions ϕ and ψ satisfy the equation

$$\begin{aligned} &\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \qquad \left(a^2 = \frac{\lambda + 2\mu}{\rho}\right) \\ &\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad \left(b^2 = \frac{\mu}{\rho}\right) \end{aligned} \tag{1.4}$$

Without loss of generality we may put $l = 1$, $b = 1$. We introduce the notation $b^2/a^2 = \gamma^2 < 1$. Then σ_{xy}, σ_{yy} may be expressed by ϕ and ψ

$$\begin{aligned} \frac{1}{\rho}\sigma_{xy} &= 2\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} \\ \frac{1}{\rho}\sigma_{yy} &= \frac{1}{\gamma^2}\frac{\partial^2\varphi}{\partial y^2} + \left(\frac{1}{\gamma^2} - 2\right)\frac{\partial^2\varphi}{\partial x^2} + 2\,\frac{\partial^2\psi}{\partial x\,\partial y} \end{aligned} \tag{1.5}$$

2. We shall tackle our problem by applying to the functions a one-sided Laplace transform with respect to t and a two-sided Laplace transform with respect to x.

We introduce the notation

$$\sigma(s, p) = \int_{-\infty}^{\infty} e^{-sx}\,dx \int_0^{\infty} e^{-pt}\sigma_{yy}(x, t)\,dt \tag{2.1}$$

$$V(s, p) = \int_{-\infty}^{\infty} e^{-sx}\, dx \int_{0}^{\infty} e^{-pt} v(x, t)\, dt \tag{2.2}$$

$$\Phi(s, p, y) = \int_{-\infty}^{\infty} e^{-sx}\, dx \int_{0}^{\infty} e^{-pt} \varphi(x, y, t)\, dt$$

$$\Psi(s, p, y) = \int_{-\infty}^{\infty} e^{-sx}\, dx \int_{0}^{\infty} e^{-pt} \psi(x, y, t)\, dt \tag{2.3}$$

$$\operatorname{Re} p > 0 \tag{2.4}$$

The question of convergence of the integrals determining these functions shall be left open for the time being. We shall deduce the relation connecting $\sigma(s, p)$ and $V(s, p)$ with the condition that $\sigma_{xy} = 0$ for $y = 0$.

From (2.3) and (1.4) we obtain

$$\frac{\partial^2 \Phi}{\partial y^2} - (\gamma^2 p^2 - s^2)\Phi = 0, \qquad \frac{\partial^2 \Psi}{\partial y^2} - (p^2 - s^2)\Psi = 0$$

The solution of these equations will be taken in the form

$$\Phi = A \exp(-y\sqrt{\gamma^2 p^2 - s^2}), \qquad \Psi = B \exp(-y\sqrt{p^2 - s^2}) \tag{2.5}$$

In the plane of the complex variable s cuts are established along the straight line passing through the points $s = p$ and $s = 0$ for the radical $\sqrt{\gamma^2 p^2 - s^2}$ from the point γp to the right to infinity and from the point $-\gamma p$ to the left to infinity, and for the positive values $\gamma^2 - s^2/p^2$ the arithmetic mean of the radical is taken. The branch of the radical $\sqrt{p^2 - s^2}$ is chosen analogously.

Using the second relation from (1.3), formulas (1.5), (2.1) to (2.5) and putting $\sigma_{xy} = 0$ for $y = 0$, we obtain

$$\frac{1}{\rho p} \frac{\sqrt{\gamma^2 - q^2}}{G(q)} \sigma(s, p) + V(s, p) = 0 \tag{2.6}$$

where

$$G(q) = (2q^2 - 1)^2 + 4q^2\sqrt{(1 - q^2)(\gamma^2 - q^2)} \qquad \left(q = \frac{s}{p}\right) \tag{2.7}$$

That is $G(q)$ is the left-hand side of Rayleigh's equation.

Formula (2.6) represents, as may be shown, the result of a double transformation of the relation, which connects the boundary values σ_{xy} and v (on the boundary $y = 0$) for the condition $\sigma_{xy} = 0$ on the boundary.

$$v = \int_0^t d\tau \int_{\xi_0}^{\xi_1} w(t - \tau, x - \xi)\, \sigma_{yy}(\xi, \tau)\, d\xi\, d\tau \qquad \left(\xi_0 = \frac{\tau - t}{\gamma} + x,\ \xi_1 = \frac{t - \tau}{\gamma} + x\right) \tag{2.8}$$

Here the kernel $w(x, t)$ is the vertical displacement of the points on the boundary $y = 0$, corresponding to the vanishing initial conditions and to the boundary conditions $\sigma_{xy} = 0$ and $\sigma_{yy} = \delta(x)\delta(t)$. In its characteristics formula (2.8) is analogous to the relation between the boundary values of a harmonic function and the normal derivative.

It can be shown that the satisfaction of (2.8) is a necessary and sufficient condition that σ_{yy} and v be the boundary values of some solution of system (1.1) in the half-plane under the condition $\sigma_{xy} = 0$ for $y = 0$.

From this it follows that (2.6) is a necessary and sufficient condition that $\sigma(s, p)$ and $v(s, p)$ are the transforms of the boundary values of some solutions of system (1.1), in the half-plane, under the condition $\sigma_{xy} = 0$ on the boundary. Therefore, (2.6) may be used to construct $\sigma(s,p)$ and $v(s, p)$ under certain conditions.

It should be pointed out that for $|x| < 1$ relation (2.8) represents an integral equation of the first kind for the determination of $\sigma_{yy}(x, t)$ and the exposition to follow below represents the construction of its solution using thereby the results of Fok [11].

3. Let us consider, as an auxiliary problem, the problem of a semi-infinite die. The initial conditions as before, shall be taken as vanishing, and on the boundary $y = 0$.

$$\sigma_{xy} = 0, \quad \sigma_{yy} = 0 \quad \text{for } x < 0, \qquad v = f(x, t) \quad \text{for } x \geqslant 0 \tag{3.1}$$

where $f(x, t)$ is a bounded function with a finite number of lines of discontinuity for $x \geqslant 0$, $t > 0$.

Using (2.6), we construct the function $\sigma_{yy}(x, t)$ for $x \geqslant 0$, $t > 0$, possessing, with respect to x, singularities not higher than $1/x$, bounded for x, and approaching infinity uniformly with respect to t, as well as the function $v(x, t)$, for $x < 0$ and $t \geqslant 0$ such that

$$v(x, t) = 0 \quad \text{for } t + \gamma x < 0 \tag{3.2}$$

and that it be integrable with respect to x in the usual sense, in an arbitrary finite interval for any t.

The requirement (3.2) means that at the front of the wave, propagating from the die, the medium is at rest, which is in accordance with the vanishing initial conditions.

If the function σ_{yy} for $x < 0$ satisfies (3.1), and for $x > 0$ possesses the properties enumerated above, then $\sigma(s, p)$, determined by (2.1), exists [10], and will be a regular function of the complex variable s for Re $s > 0$ and will approach 0 as s approaches infinity.

In order to clarify the question of existence of $V(s, p)$ from (2.2), it will be represented in the form

$$V(s,p) = V_1(s,p) + V_2(s,p) \tag{3.3}$$

where

$$V_1(s,p) = \int_0^\infty e^{-sx}dx \int_0^\infty v(x,t)e^{-pt}dt, \qquad V_2(s,p) = \int_{-\infty}^0 e^{-sx}dx \int_0^\infty v(x,t)\,e^{-pt}dt$$

From what was said above with regard to the properties of $f(x, t)$ and from (3.1), it follows that $V_1(s, p)$ will be regular for Re $s > 0$ and will approach 0 at infinity [10], at least as s^{-1}. From (3.2), in accordance with [10], it follows that $V_2(s, p)$ will be regular for Re $s < \gamma$ Re p and will also approach 0 at infinity.

Thus the determination of the functions $\sigma(s, p)$ and $V_2(s, p)$ (the function V_1 is known from condition (3.1)) is reduced to the following problem. To find the function $\sigma(s, p)$, regular for Re $s > 0$ and approaching 0, as s approaches infinity; and the function $V_2(s, p)$, regular for Re $s < \gamma$ Re p, and also approaching 0 as s approaches infinity in such a manner that

$$\frac{1}{p}\frac{\sqrt{\gamma^2-q^2}}{G(q)}\sigma(s,p) + V_1(s,p) = -V_2(s,p) \qquad \left(q=\frac{s}{p}\right) \tag{3.4}$$

with

$$\text{Re } p > 0, \qquad 0 < \text{Re} s < \gamma \, \text{Re}\, p \tag{3.5}$$

Here $V_1(s, p)$ is a known function, regular for Re $s > 0$ and becoming 0 at infinity as s^{-1}.

This problem will be solved by the method advanced in paper [11].

We represent the coefficient of $\sigma(s, p)$ from (3.4) in the form

$$\frac{1}{\rho p}\frac{\sqrt{\gamma^2-q^2}}{G(q)} = -\frac{1}{2\rho(1-\gamma^2)p}K_1(q)K_2(q) \tag{3.6}$$

Here

$$K_1(q) = \frac{\sqrt{\gamma+q}}{q+\vartheta}\exp[-g_1(q)], \qquad K_2(q) = \frac{\sqrt{\gamma-q}}{q-\vartheta}\exp[-g_2(q)] \tag{3.7}$$

$$g_1(q) = \frac{1}{\pi}\int_{-1}^{-\gamma}\varphi(\zeta)\frac{d\zeta}{\zeta-q}, \qquad g_2(q) = -\frac{1}{\pi}\int_{\gamma}^{1}\varphi(\zeta)\frac{d\zeta}{\zeta-q} \tag{3.8}$$

$$\varphi(\zeta) = \text{arc tg}\,\frac{4\zeta^2\sqrt{(\zeta^2-\gamma^2)(1-\zeta^2)}}{(2\zeta^2-1)^2} \tag{3.9}$$

thereby $1/\theta$ is the root of Rayleigh's equation $G(1/g) = 0$.

Using the properties of Cauchy type integral [12], we conclude that

$K_1(q)$ is a regular function of the complex variable q outside the interval $-1 \leqslant q \leqslant -\gamma$, and does not have there any zeros except at $q = \infty$, $K_2(q)$ is regular outside the interval $\gamma \leqslant q \leqslant 1$ and also does not possess there any zeros, except at $q = \infty$. At infinity K_1 and K_2 approach zero as $q^{-1/2}$. Then $K_1(s/p)$ is regular outside the interval connecting the points $-\gamma p$ and $-p$, and $K_2(s/p)$ is outside the interval $[\gamma p, p]$ and also does not possess any zeros at finite points of the plane outside of these intervals and becomes 0 at infinity as $s^{-1/2}$. Using (3.6) we rewrite (3.4) in the following manner:

$$\frac{1}{2\rho(1-\gamma^2)p} K_1(q)\,\sigma(s,\ p) - \frac{V_1(s,p)}{K_2(q)} = \frac{V_2(s,p)}{K_2(q)} \qquad (3.10)$$

$$q = s/p, \qquad 0 < \operatorname{Re} s < \gamma \operatorname{Re} p$$

The second term on the left-hand side of (3.10) is obviously regular within the strip $0 < \operatorname{Re} s < \gamma \operatorname{Re} p$ and tends to 0 as $s^{-1/2}$ when s approaches infinity. Therefore, using Cauchy type integrals, we may represent it in the form

$$\frac{V_1(s,p)}{K_2(q)} = L_1(s,\ p) + L_2(s,p)$$

where

$$L_1(s,p) = -\frac{1}{2\pi i}\int_{l_1} \frac{V_1(\xi,p)\,d\xi}{K_2(\xi/p)(\xi - s)}, \qquad L_2(s,p) = \frac{1}{2\pi i}\int_{l_2} \frac{V_1(\xi,p)\,d\xi}{K_2(\xi/p)(\xi - s)} \qquad (3.11)$$

Here the contours l_1 and l_2 are rectilinear, parallel to the imaginary axis, and are situated along the left and right edge of the strip $0 <$ · $\operatorname{Re} s < \gamma \operatorname{Re} p$, respectively. It is not difficult to note that $L_1(s,\ p)$ is regular for $\operatorname{Re} s > 0$ and approaches 0 at infinity, while $L_2(s,\ p)$ is regular for $\operatorname{Re} s < \gamma \operatorname{Re} p$ and also approaches 0 at infinity.

We rewrite (3.10) in the following manner:

$$\frac{1}{2\rho(1-\gamma^2)p} K_1(q)\,\sigma(s\ p) - L_1(s,p) = L_2(s,p) + \frac{V_2(s,p)}{K_2(q)} \qquad (3.12)$$

As was indicated above, each term on the left and right-hand sides of (3.12) is regular at $\operatorname{Re} s > 0$ and $\operatorname{Re} s < \gamma \operatorname{Re} p$ respectively, and approaches 0 at infinity. Then, from the satisfaction of (3.12) on the strip (3.5) it follows that (3.12) is satisfied on the whole plane s and from this, taking into account the properties of the terms, on the basis of Liouvilles' theorem, we conclude that the left- and right-hand sides of (3.12) are identically equal to zerō. Then

$$\sigma(s,p) = 2\rho(1-\gamma^2)\,p\,\frac{L_1(s,p)}{K_1(q)}, \qquad V_2(s,p) = -K_2(q)\,L_2(s,p) \quad (3.13)$$

where $q = s/p$ and K_1, K_2, L_1, L_2 are given by formulas (3.7) and (3.11).

It is not difficult to verify that the functions $\sigma(s, p)$ and $V_2(s, p)$ so constructed satisfy the requirements of regularity and behavior at infinity formulated for them. The uniqueness of the solution of the problem studied in this section is easily proved, considering the corresponding homogeneous problem.

In order to conclude the solution of the problem of the "semi-infinite die", the functions $\sigma(s, p)$ and $V_2(s, p)$ should be subjected to inverse transformation and $\sigma_{yy}(x, t)$ should be constructed for $x \geqslant 0$, and $v(x, t)$ should be constructed for $x < 0$.

4. We shall consider first the most simple case when $f(x, t)$ from (3.2) does not depend on x, that is, $f(x, t) = f(t)$. We have

$$V_1(\xi, p) = \frac{F(p)}{\xi}, \quad F(p) = \int_0^\infty f(t) e^{-pt}\, dt$$

Formula (3.13) takes on the form:

$$\sigma(s, p) = -\frac{2\rho(1-\gamma^2)\vartheta}{\sqrt{\gamma}} pF(p) \frac{\exp[g_2(0)]}{sK_1(q)}, \quad V_2(s, p) = -\left[1 + \frac{\vartheta}{\sqrt{\gamma}} K_2(q)\right] \frac{F(p)}{s}$$

Performing inverse transforms we obtain

$$\sigma_{yy}(x, t) = -\frac{\rho}{\gamma} f'(t) \qquad \text{for } t - \gamma x < 0$$

$$\sigma_{yy}(x, t) = -\frac{\rho}{\gamma} f'(t) + \frac{\sqrt{2(\gamma^{-2}-1)}}{\pi} \frac{\partial}{\partial t} \int_\gamma^{t/x} \frac{(\vartheta - \xi) A(\xi)}{\xi \sqrt{\xi - \gamma}} f(t - x\xi)\, d\xi \qquad \text{for } t - \gamma x > 0 \tag{4.1}$$

$$v(x, t) = 0 \qquad \text{for } t + \gamma x < 0$$

$$v(x, t) = \frac{1}{\pi\sqrt{2\gamma(1-\gamma^2)}} \int_\gamma^{-t/x} \frac{\sqrt{\xi - \gamma}\, B(\xi)}{\xi(\xi - \vartheta)} f(t + \xi x)\, d\xi \qquad \text{for } t + \gamma x > 0 \tag{4.2}$$

Here

$$\begin{aligned} &A(\xi) = \exp[g_1(-\xi)], \qquad B(\xi) = \exp[-g_2(\xi)] \qquad \text{for } \xi \geqslant 1 \\ &A(\xi) = \cos\varphi(\xi) \exp[g_1(-\xi)], \qquad B(\xi) = \cos\varphi(\xi) \exp[-g_2(\xi)] \qquad \text{for } \xi < 1 \end{aligned} \tag{4.3}$$

Here $g_1(-\xi)$ and $g_2(\xi)$ for $\xi > 1$ are given by formulas (3.8) and for $\gamma < \xi \leqslant 1$ by the same formulas, where the integrals are taken in the sense of their principal value. $\phi(\xi)$ is given by formula (3.9). In (4.2) the integral is also taken in the sense of its principal value. A study of the physical consequences will be presented further below.

5. Let us consider now the general case of the problem of the "semi-infinite die". It is not difficult to verify that (3.13) may be transformed to the form

$$\sigma(s,p) = -\rho p \frac{G(q)}{\sqrt{\gamma^2-q^2}} V_1(s,p) + \frac{2\rho(1-\gamma^2)p}{K_1(q)} \frac{1}{\pi} \int_\gamma^\infty \frac{(\xi-\vartheta)C(\xi)}{\sqrt{\xi-\gamma}(\xi-q)} V_1(\xi p, p)\,d\xi$$

It is not difficult to notice that the first term in this formula is a double Laplace transform (one-sided with respect to t and two-sided with respect to x) of the value of $\sigma_{yy}(x, t)$ for $y = 0$, corresponding to the condition for $y = 0$

$$\sigma_{xy} = 0, \quad v = 0 \quad \text{for } x < 0, \qquad v = f(x,t) \quad \text{for } x > 0$$

These conditions lead us to the solved boundary value problem [7], which permits to construct the value of σ_{yy}, of interest to us, and which shall be designated by $\sigma_{yy}{}^0(x, t)$. Performing inverse transformations also of the second term, we obtain

$$\sigma_{yy}(x,t) = \sigma_{yy}{}^0(x,t) \qquad \text{for } t-\gamma x < 0$$

$$\sigma_{yy}(x,t) = \sigma_{yy}{}^0(x,t) - \frac{2(1-\gamma^2)}{\pi^2} \frac{\partial^2}{\partial t^2} \int_0^{\tau_0} d\tau \int_0^{\zeta_0} R_1(\zeta_1\ x/\zeta)\, v(\zeta,\tau) \frac{d\zeta}{\zeta} \qquad (5.1)$$

$$\text{for } t-\gamma x > 0$$

Performing inverse transformations of the function $V_2(s, p)$ from (3.13) we obtain

$$v(x,t) = 0 \qquad \text{for } t+\gamma x < 0$$

$$v(x,t) = \frac{1}{\pi^2} \frac{\partial}{\partial t} \int_0^{\tau_0} d\tau \int_0^{\zeta_0} R_2(\zeta_1, x/\zeta)\, v(\zeta,\tau) \frac{d\zeta}{\zeta} \qquad \text{for } t+\gamma x > 0 \qquad (5.2)$$

In formulas (5.1) and (5.2) the following notation has been introduced

$$\tau_0 = t - \gamma|x|, \qquad \zeta_0 = \frac{\tau_0-\tau}{\gamma}, \qquad \zeta_1 = \frac{\tau_0-\tau}{\zeta}.$$

$$R_1(\zeta_1, x/\zeta) = \sqrt{\frac{\zeta}{x}} \int_\gamma^{\zeta_1} \frac{F_1(\xi)C(\xi)DA(\eta)}{F_2(\xi,\zeta_1)[\zeta(\zeta_1-\xi)-x(\xi+\gamma)]}\,d\xi$$

$$R_2(\zeta_1, x/\zeta) = \zeta\sqrt{-x\zeta} \int_\gamma^{\zeta_1} \frac{F_1(\xi)F_2(\xi,\zeta_1)C(\xi)B(\eta)}{D[\zeta(\zeta_1-\xi)+x(\xi-\gamma)]}\,d\xi$$

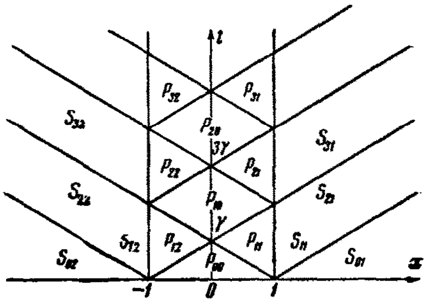
Here

$$\eta = \frac{t-\tau-\zeta\xi}{x}, \quad F_1(\xi) = \frac{\xi-\vartheta}{\sqrt{\xi-\gamma}}, \quad F_2(\xi,\zeta_1) = \sqrt{\zeta_1-\xi}$$

$$D = \zeta(\zeta_1-\xi) - |x|(\vartheta-\gamma)$$

The functions $A(\xi)$ and $B(\xi)$ are given by formulas (4.3). It is not difficult to verify that the constructed functions $\sigma_{yy}(x, t)$ and $v(x, t)$ satisfy all the requirements which we imposed upon them (see Section 3).

6. We shall show now how the values of σ_{yy} for $|x| < 1$ and $v(x, t)$ for $|x| > 1$, may be constructed from conditions (1.2) and using the solution of the problem of the "semi-infinite die". The line of thought is analogous to the one used in paper [9]. We consider the plane $(x, t)y = 0$. By virtue of vanishing initial conditions in the regions S_{01} and S_{02} $v(x, t) = 0$.



In regions S_{11} and S_{12} we can construct the values of $v(x, t)$ using the solution of the problem of a semi-infinite die, since in formula (5.2), transformed to the new system of coordinates, the integration will be carried out within the region $P_{00} + P_{00}$ or $P_{00} + P_{12}$, i.e. there, where the values of $v(x, t)$ are given.

In region $P_{00}\sigma_{yy}(x, t)$ will be constructed by a well-known method, i.e. as the normal stress component, corresponding to $\sigma_{xy} = 0$ and to the given vertical displacement [7]. In regions P_{11} and P_{12}, $\sigma_{yy}(x, t)$ is constructed with the aid of the solution of the problem of the "semi-infinite die", since in formula (5.1), transformed to the new system of coordinates, the integration will be carried out within the regions $P_{00} + P_{12}$ or $P_{00} + P_{12}$, where the values $v(x, t)$ are given. The values of $v(x, t)$ in regions $S_{21} + S_{22}$ and the values of $\sigma_{yy}(x, t)$ in regions P_{10}, P_{21}, P_{22} are constructed in the same manner as in regions S_{11}, S_{12} and P_{00}, P_{11}, P_{12}, respectively, since the integration must be carried out in the regions lying within $P_{21} + P_{10} + P_{11} + P_{12} + P_{00} + S_{02} + S_{12}$ or $P_{22} + P_{10} + P_{11}$ $P_{12} + P_{00} + S_{01} + S_{11}$, where the values of $v(x, t)$ will already be known.

Continuing this process, it is obviously possible to construct the values of $\sigma_{yy}(x, t)$ for $|x| < 1$ and $v(x, t)$ for $|x| > 1$, which concludes the solution of the problem.

7. Let us investigate some properties of the solution of the problem of the "semi-infinite die". We shall consider first the expression for the vertical displacement on the free part of the boundary $x < 0$, which is given by (4.2).

As we required, $v = 0$ for $t + \gamma x < 0$, which in accordance with the vanishing initial condition $t + \gamma x = 0$ is the equation of the front of the longitudinal waves, propagating from the die. At the front

$$v = 0, \quad \frac{\partial v}{\partial t} = 0, \quad \frac{\partial v}{\partial x} = 0$$

It is not difficult to verify that $v(x, t)$ is continuous at the point $x = 0$, however, $\partial v/\partial x$ for x approaching 0, is not bounded from the left and may be represented for small values of x in the form

$$\frac{\partial v}{\partial x} = \frac{1}{\pi \sqrt{2\gamma(1-\gamma^2)}} \left[f(0) t^{-1/2} + \int_0^t \frac{f(\tau)\, d\tau}{\sqrt{t-\tau}} \right] \frac{1}{\sqrt{-x}}$$

Using [12], Section 29, we conclude that $v(x, t)$ for $t/x = -\theta$ possesses a logarithmic singularity, i.e. there is a logarithmic discontinuity in the vertical displacement on the free part of the boundary, which is propagated with the speed of Rayleigh waves, (see Section 3), as could have been expected *a priori*. Investigating expressions (4.2) and (4.3), the front of longitudinal waves $t + x = 0$ may also be separated, however, just as at the front of the longitudinal waves, $v(x, t)$ and both of its first derivatives with respect to x and t, will be continuous.

The most simple case occurs when $f(t) = 1$, $t > 0$, then

$$v(x, t) = \frac{1}{\pi \sqrt{2\gamma(1-\gamma^2)}} \int_{\gamma}^{-t/x} \frac{\sqrt{\xi - \gamma}}{\xi(\xi - v)} B(\xi)\, d\xi$$

It is important to emphasize that in this case $v(x, t)$ is a homogeneous function of order zero in the variables x and t, and this means that in this case the problem of the semi-infinite die could have solved by the method of functional invariant solutions of Smirnov and Sobolev [4]. Let us study now the properties of $\sigma_{yy}(x, t)$. From (4.1)we have

$$\sigma_{yy}(x, t) = -\frac{p}{\gamma} f'(t), \quad t - \gamma x < 0$$

i.e. at the points which were not reached as yet by the disturbance, "reflected" from the free part of the boundary, the stress is the same which would be obtained if the condition $\sigma_{xy} = 0$, $v = f(t)$ were prescribed along the whole boundary, as could be expected.

At instant $t = \gamma x$ the front of the longitudinal wave reaches the point x; at the front $\sigma_{yy} = 0$ if $f(0) = 0$ and $\sigma_{yy} \to \infty$ as $a(x)/\sqrt{t - \gamma x}$ if $f(0) \neq 0$; thereby, as x approaches infinity, $a(x)$ approaches 0 as x^{-1}.

Studying (4.1) and (4.3) we also may separate the front of the propagating transverse wave $t - x = 0$. Underneath the die $(x > 0)$, precisely such conditions are obtained in this case, when the propagating longitudinal or transverse waves, in interaction with the boundary, produces only the longitudinal or the transverse wave respectively [7]. Thus, for

$t < x < t/\gamma$ the stress σ_{yy} from (4.1) is the stress in the longitudinal wave, and for $0 < x < t$ the total stress in the longitudinal and transverse waves.

At the points $x = 0$, just as in the solution in the corresponding static problem, there occurs an integrable singularity for small x, namely

$$\sigma_{yy}(x, t) = -\frac{\sqrt{2(1/\gamma^2 - 1)}}{\pi}\rho\frac{\partial}{\partial t}\int_0^t \frac{f(\tau)\,d\tau}{\sqrt{t-\tau}}\frac{1}{\sqrt{x}}$$

The investigation of the general case of the problem of the "semi-infinite die" shows that if $f(x, t) = 0$ from (3.2) for $x = 0$, then $\sigma_{yy}(x, t)$ and $\partial v/\partial x$ will be continuous at the point $x = 0$, in the corresponding static case we have the same picture.

It is important to emphasize that the problem of the die investigated in the present paper could be also solved by the method of functional-invariant solutions of Smirnov and Sobolev in the following manner.

Considering the problem of the "semi-infinite die" we must put $f(x, t) = 1$ for $x > \eta$ and $t > \tau$, where η, $\tau > 0$ are arbitrary parameters and $f = 0$ for other values of x and t. $\sigma_{yy}(x, t)$ may be constructed by the functional-invariant method valid for $t - \tau < \gamma\eta$. The obtained wave should be represented in the form of superposition of plane waves, and after this, one should study how each wave will interact with the free boundary, and the result should then again be added.

Summing up the values of σ_{yy}, corresponding to different values of η and τ, one can obtain the value of σ_{yy}, corresponding to an arbitrary function $f(x, t)$. The passage from the semi-infinite die to a finite die may be carried out in the same manner as in the present paper.

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